ISRAEL JOURNAL OF MATHEMATICS, Vol. 23, Nos. 3–4, 1976

ON COMPLEMENTED SUBSPACES OF $(\Sigma\, l_2)_{l_p}$

BY

E. ODELL[†]

ABSTRACT

It is shown that if X is a complemented subspace of $(\Sigma\, l_2)_{l_p}$ $(1 < p < \infty)$, then X is isomorphic to either l_2, l_p, $l_2 \oplus l_p$ or $(\Sigma\, l_2)_{l_p}$. If X is a complemented subspace of C_p $(1 < p < \infty)$ which does not contain an isomorph of $(\Sigma\, l_2)_{l_p}$ then X is isomorphic to a complemented subspace of $(\Sigma\, C_p^n)_{l_p} \oplus l_2$.

0. Introduction

We prove (Theorem 1) that if X is a complemented infinite dimensional subspace of $Z_p = (\Sigma\, l_2)_{l_p}$ $(1 < p < \infty)$, then X is isomorphic to one of the four spaces: l_2, l_p, $l_2 \oplus l_p$ or Z_p.

Pelczynski [8] has shown that every complemented infinite dimensional subspace of l_p is isomorphic to l_p and Edelstein and Wojtaszczyk [3] proved that a complemented infinite dimensional subspace of $l_p \oplus l_2$ is isomorphic to either l_2, l_p or $l_p \oplus l_2$. Our result is thus a continuation of the work of these people (and many others) in the study of the isomorphic structure of complemented subspaces of L_p.

It should be mentioned that G. Schechtman [11] has obtained our main result under the additional assumption that X has an unconditional basis. He has also exhibited the complex structure of Z_p by exhibiting an infinite number of mutually non-isomorphic complemented subspaces of L_p, all of which embed isomorphically into Z_p [10].

The proof of the main result (Theorem 1) is given in Section 2. Section 1 introduces the notation which we employ. Unfortunately, while the ideas are not difficult, notation necessary to present the proof of Theorem 2 is somewhat complicated. We urge the interested reader to view Z_p as an infinite matrix space and construct his own diagrams and pictures as he proceeds.

[†] This research was partially supported by NSF MPS 72-04634-A03.
Received November 21, 1975

Section 3 is an extension of the techniques of Section 2 to the Banach space C_p of compact operators x on l_2 with $\|x\| = (\text{trace}(x^*x)^{p/2})^{1/p}$. The result proved (Theorem 3) is that if X is a complemented subspace of C_p $(1 < p < \infty)$ which contains no isomorph of Z_p then X is isomorphic to a complemented subspace of $(\Sigma C_p^n)_{l_p} \oplus l_2$.

We wish to thank Professors W. B. Johnson and P. Wojtaszczyk for many useful discussions regarding the material contained herein. In particular the derivation of Theorem 1 from Theorem 2 is due to Professor Wojtaszczyk and we wish to thank him for allowing us to reproduce here his proof.

1. Notation

We let $(e_{ij})_{i,j=1}^{\infty}$ be the natural basis for Z_p. Thus Z_p is the Banach space of all scalars $(a_{ij})_{i,j=1}^{\infty}$ such that

$$\left\| \sum_j \sum_i a_{ij} e_{ij} \right\| = \left(\sum_j \left(\sum_i a_{ij}^2 \right)^{p/2} \right)^{1/p} < \infty .$$

If n is any integer, Q_n denotes the natural projection of Z_p onto the first n Hilbert spaces:

$$Q_n\left(\sum_j \sum_i a_{ij} e_{ij} \right) = \sum_{j=1}^{n} \sum_{i=1}^{\infty} a_{ij} e_{ij} .$$

We let I be the identity operator on Z_p and $Q^n = I - Q_n$ is the natural projection onto those Hilbert spaces past the first n.

For any integer l we define O_l by

$$O_l\left(\sum_j \sum_i a_{ij} e_{ij} \right) = \sum_j \sum_i a_{ij} e_{ij} - \sum_{j=1}^{l} \sum_{i=1}^{l} a_{ij} e_{ij} .$$

This is the projection which restricts the support of a vector to those e_{ij} which lie "outside" the initial l by l block of the basis.

If $n < m$, let

$$Q_{n,m} = Q_m - Q_n ; \quad O_{n,m} = O_m - O_n .$$

Finally by a *staircase mapping* R we mean an operator of the form

$$R = O_{l_1} Q_{k_0, k_1} + O_{l_2} Q_{k_1, k_2} + O_{l_3} Q_{k_2, k_3} + \cdots$$

where $(l_i)_{i=1}^{\infty}$ and $(k_i)_{i=1}^{\infty}$ are increasing sequences of positive integers.

It is easy to see that all of the above operators are norm 1 projections on Z_p. Note that the range of Q_n is isomorphic to l_2 and the kernel of a staircase mapping R is isomorphic to $l_2 \oplus l_p$. The latter follows from the fact that if (s_i) is any sequence of integers, then $(\Sigma l_2^{s_i})_{l_p}$ is isomorphic to a complemented subspace of l_p and hence to l_p [8].

X will always refer to an infinite dimensional Banach space. We use standard Banach space notation throughout and any terms or expressions not defined above may be found in the book of Lindenstrauss and Tzafriri [7].

2. The main result

THEOREM 1. *Let X be a complemented subspace of Z_p $(1 < p < \infty)$. Then X is isomorphic to one of the four spaces l_p, l_2, $l_p \oplus l_2$* or *Z_p.*

Theorem 1 follows from Theorem 2 and the results of [1].

THEOREM 2. *Let X be a subspace of Z_p $(2 < p < \infty)$ which does not contain an isomorph of Z_p and let T be a projection of Z_p onto X. Then for all $\varepsilon > 0$ there exists an integer N and a staircase mapping R such that if $y \in Q^N Z_p$ then $\| RTy \| \leqq \varepsilon \| y \|$.*

Proof of Theorem 1 from Theorem 2

By duality we may assume $p > 2$ (the theorem is trivial for $p = 2$). If X contains an isomorph of Z_p then by theorem 2.1 of [1] it contains a complemented isomorph of Z_p and thus by the decomposition method of Pelczynski [8], X is isomorphic to Z_p. We may therefore assume that X satisfies the hypothesis of Theorem 2.

Let $0 < \varepsilon < 1/2(1 + \| T \|)$ and let R and N be as in the conclusion of Theorem 2. R is a mapping of the form

$$R = O_{l_1} Q_{n_0, n_1} + O_{l_2} Q_{n_1, n_2} + \cdots .$$

By considering the maximum of N and n_0 and redefining R if necessary we may assume that $n_0 = N$.

Let $S = I - RT$, and note that $S(X) \subseteq \operatorname{Ker} R$ which, as observed above, is isomorphic to $l_p \oplus l_2$. We shall show that S is an isomorphism of Z_p onto Z_p. But then the restriction of STS^{-1} to $\operatorname{Ker} R$ is a projection of $\operatorname{Ker} R$ onto $S(X)$. Thus by [3], $S(X)$ and hence X itself is isomorphic to l_p, l_2 or $l_p \oplus l_2$.

To see that S is an isomorphism of Z_p onto Z_p, write

$$S = I - RT = I - RTQ^N - RTQ_N .$$

Since $(RTQ_N)^2 = 0, I - RTQ_N$ is an isomorphism of Z_p onto Z_p with inverse $I + RTQ_N$. By Theorem 2,

$$\| RTQ^N \| < \varepsilon < 1/(2\| I + RTQ_N \|)$$

and thus $I - RTQ_N - RTQ^N = S$ is an isomorphism of Z_p onto Z_p. **Q.E.D.**

The rest of this section is devoted to the proof of Theorem 2. We begin with some elementary remarks:

Let (y_k) be a block basis of (e_{ij}) in Z_p $(2 < p < \infty)$ and let (a_i) be a finitely non-zero sequence of scalars. Then

$$\left\| \sum a_k y_k \right\| \leqq \left(\sum | a_k |^2 \| y_k \|^2 \right)^{1/2}. \tag{2.1}$$

If in addition $\| Q_{j-1,j} y_k \| = \| Q_{j-1,j} y_l \|$ for all j, k and l then (cf. [10, p. 292])

$$\left\| \sum a_k y_k \right\| = \left(\sum | a_k |^2 \| y_k \|^2 \right)^{1/2}. \tag{2.2}$$

Our first lemma was essentially proved in [1]. Its importance to us is that our key lemma (Lemma 2) is derived from it.

LEMMA 1. *Let X be a subspace of Z_p $(2 < p < \infty)$ and assume that for every integer n and $K > 0$ there is a normalized block basis (z_i) of (e_{ij}) with $z_i \in X$ for all i and such that (z_i) is 2-equivalent to the unit vector basis of l_2. Assume also that*

$$\sup_i \| Q_n z_i \| < 1/K. \tag{2.3}$$

Then X contains an isomorph of Z_p.

PROOF. If $z = \Sigma a_i z_i \in X$ *then by* (2.1) *and* (2.3)

$$\| Q_n z \| = \left\| \sum a_i Q_n z_i \right\| \leqq \left(\sum | a_i |^2 \| Q_n z_i \|^2 \right)^{1/2}$$

$$\leqq \frac{1}{K} \left(\sum | a_i |^2 \right)^{1/2} \leqq \frac{2}{K} \| z \|.$$

Thus the entire span of the z_i "sits" almost entirely in $Q^n Z_p$. The proof of Theorem 2.1 of [1] yields the lemma. **Q.E.D.**

LEMMA 2. *Let X and T be as in the statement of Theorem 2. Then there is an integer n and a $K < \infty$ so that for all integers $m > n$ and $\delta > 0$ there is an integer l such that if $y \in Q_l Z_p$ and $\| y \| \leqq 1$ then either*

$$\| Q_{n,m}Ty \| \leqq \delta \tag{2.4}$$

or

$$\| Q_{n,m}Ty \| \leqq K \| Q_nTy \| . \tag{2.5}$$

PROOF. We shall show that if n and K fail the conclusion of the lemma, then there is a normalized block basis $(z_i) \subseteq X$ which is 2-equivalent to the unit vector basis of l_2 and such that $\| Q_nz_i \| \leqq 1/K$ for all i. Since X contains no isomorph of Z_p, Lemma 1 implies that some n and K must satisfy the conclusion of the lemma.

Thus let n and K be fixed and assume that there is an $m > n$ and a $\delta > 0$ such that no l satisfies the conclusion of the lemma. Then there is a sequence (y_i) in Z_p converging weakly to 0 with $\| y_i \| \leqq 1$ and so that for all i, y_i fails both (2.4) and (2.5).

Let $x_i = Ty_i / \| Ty_i \|$. Then (x_i) is a weakly null sequence in X satisfying for all i,

$$\| Q_{n,m}x_i \| > \delta / \| T \| \tag{2.6}$$

and

$$\| Q_{n,m}x_i \| > K \| Q_nx_i \| . \tag{2.7}$$

By passing to a subsequence if necessary we may assume that (x_i) is a normalized block basis of (e_{ij}), which by virtue of (2.6) (and $p > 2$) is equivalent to the unit vector basis of l_2. We may also assume (again by passing to a subsequence and perturbing slightly) that if $n < j \leqq m$ then for all i and k,

$$\| Q_{j-1,j}x_i \| = \| Q_{j-1,j}x_k \| . \tag{2.8}$$

By lemma 2.4 of [1] there is a normalized block basis (z_i) of (x_i) such that (z_i) is 2-equivalent to the unit vector basis of l_2. For each i let

$$z_i = \sum_{k \in A_i} \alpha_k x_k$$

where (A_i) is a sequence of finite disjoint subsets of $\mathbf{N}$.

By (2.1), (2.2), (2.7) and (2.8),

$$\| Q_nz_i \| \leqq \left(\sum_{k \in A_j} \| Q_n\alpha_k x_k \|^2 \right)^{1/2} \leqq \frac{1}{K} \left(\sum_{k \in A_i} \alpha_k^2 \| Q_{n,m}x_k \|^2 \right)^{1/2}$$

$$= \frac{1}{K} \| Q_{n,m}z_i \| \leqq \frac{1}{K} .$$

Q.E.D.

Our next lemma, which is quite elementary, will be used below to choose N in the proof of Theorem 2.

LEMMA 3. *Let T be an operator on Z_p where $p > 2$. Then for all $\alpha > 0$ and every integer n there is an integer $N = N(\alpha, n)$ so that if $x \in Q^N Z_p$ then*

$$\| Q_n T x \| \leqq \alpha \| x \|. \tag{2.9}$$

PROOF. If not then there is an $\alpha > 0$ and an integer n so that for all N there is an $x_N \in Q^N Z_p$ with $\|x_N\| \leqq 1$ and

$$\| Q_n T x_N \| > \alpha. \tag{2.10}$$

Clearly there is a subsequence (x_{N_i}) of (x_N) which is equivalent to the unit vector basis of l_p. Since $(T x_{N_i})$ converges weakly to 0 we may assume $(T x_{N_i})$ is a block basis of (e_{ij}). But then (2.10) implies that $(T x_{N_i})$ is equivalent to the unit vector basis of l_2. This is impossible since $p > 2$. **Q.E.D.**

We are now ready to produce the desired staircase mapping R. For the sake of clarity we first state one more lemma.

LEMMA 4. *Let T be an operator on Z_p $(1 < p < \infty)$ and let $(\delta_i)_{i=0}^{\infty}$ be a given sequence of positive numbers. Then there exist integers $0 = u_0 < u_1 < \cdots$ and $0 = v_0 < v_1 < \cdots$ so that if $E_i = O_{u_i, u_{i+1}} Z_p$ then for all $x \in E_i$ with $\| x \| \leqq 1$ we have*

$$\| O_{v_{i+1}} T x \| + \| (I - O_{v_{i-1}}) T x \| \leqq \delta_i. \tag{2.11}$$

This holds for all $i \geqq 0$ with the convention that $O_0 = O_{-1} = I$.

PROOF. Let $u_0 = v_0 = 0$ and let u_1 be arbitrary. Since the unit ball of $(I - O_{u_1}) Z_p$ is compact we can find v_1 so that (2.11) holds for $i = 0$. Then choose $u_2 > u_1$ so that if $x \in O_{u_2} Z_p$ with $\| x \| \leqq 1$ then

$$\| (I - O_{v_1}) T x \| < \delta_2 / 2.$$

This may be done since if $x_j \in O_j Z_p$ and $\| x_j \| \leqq 1$ then $T x_j$ converges weakly to 0. By compactness again choose $v_2 > v_1$ so that if $x \in O_{u_1, u_2} Z_p$ and $\| x \| \leqq 1$ then

$$\| O_{v_2} T x \| < \delta_1.$$

Continuing in this way we can inductively construct (u_i) and (v_i) so that (2.11) holds for all i. **Q.E.D.**

Proof of Theorem 2

Let X and T be as in the statement of Theorem 2 and let $\varepsilon > 0$. Let n and K be as given in the conclusion of Lemma 2.

Since $Q_n Z_p$ is isomorphic to l_2, there is a constant $c(n) < \infty$ so that if (y_i) is a normalized block basis of (e_{ij}) in $Q_n Z_p$ then (y_i) is $c(n)$-equivalent to the unit vector basis of l_2.

Choose $\alpha > 0$ so that

$$Kc(n)\alpha < \varepsilon / 14. \tag{2.12}$$

Let $N = N(\alpha, n)$ satisfy the conclusion of Lemma 3 and let $(\delta_i)_{i=0}^{\infty}$ be any sequence of positive numbers such that $\delta_0 < \alpha/2$ and for all $j \geqq 1$,

$$\sum_{i=j}^{\infty} \delta_i < \delta_{j-1}. \tag{2.13}$$

Finally let (u_i), (v_i) and (E_i) be given by Lemma 4.

We shall construct staircase mappings R_1 and R_2 so that if $y \in [E_{2i}]_{i=0}^{\infty} \cap Q^N Z_p$ (respectively $y \in [E_{2i-1}]_{i=1}^{\infty} \cap Q^N Z_p$) and $\| y \| \leqq 1$ then

$$\| R_1 T y \| < \varepsilon / 2 \tag{2.14}$$

(respectively $\| R_2 T y \| < \varepsilon/2$). Theorem 2 follows by letting R be any staircase mapping such that

$$RZ_p \subseteq R_1 Z_p \cap R_2 Z_p.$$

Without loss of generality we may assume that $v_1 > n$. Repeated application of Lemma 2 (for $m = v_{2j-1}$ and $\delta = \delta_j$) yields for each j an *odd* integer s_j which satisfies the following condition:

If $x \in [E_i]_{i=s_j}^{\infty}$ with $\| x \| \leqq 1$ then either

$$\| Q_{n, v_{2j-1}} T x \| \leqq \delta_j \tag{2.15}$$

or

$$\| Q_{n, v_{2j-1}} T x \| \leqq K \| Q_n T x \|. \tag{2.16}$$

Clearly we may assume $s_1 < s_2 < \cdots$ and we define R_1 by

$$R_1 = O_{v_{s_1}} Q_{n, v_1} + O_{v_{s_2}} Q_{v_1, v_3} + O_{v_{s_3}} Q_{v_3, v_5} + \cdots. \tag{2.17}$$

We check that R_1 satisfies (2.14). Let $y = \sum_{j=0}^{\infty} y_j \in [E_{2i}]_{i=0}^{\infty} \cap Q^N Z_p$, $\| y \| \leqq 1$, where for each $j \geqq 0$

$$y_i \in [E_i]_{i=s_j+1}^{s_{j+1}-1} = O_{u_{s_j+1}, u_{s_{j+1}}} Z_p. \tag{2.18}$$

(We let $s_0 = -1$ and recall that $E_i = O_{u_i, u_{i+1}} Z_p$.)

Now $\| y_j \| \leqq 1$ for each j and so by (2.11), (2.13) and (2.18)

$$\| (I - O_{v_{s_j}}) T y_j \| + \| O_{v_{s_{j+1}}} T y_j \| \leqq \delta_{s_j} \leqq \delta_j. \tag{2.19}$$

Let us assume for the moment that the expression in (2.19) equals 0 so that

$$T y_j \in O_{v_{s_j}, v_{s_{j+1}}} Z_p \quad \text{for all} \quad j. \tag{2.20}$$

Now by (2.17) and (2.20),

$$R_1 T y_j = R_1 O_{v_{s_j}, v_{s_{j+1}}} T y_i = Q_{n, v_{2j-1}} T y_j. \tag{2.21}$$

By (2.15), (2.16), (2.20) and (2.21) for each j either

$$\| R_1 T y_j \| \leqq \delta_j \tag{2.22}$$

or

$$\| R_1 T y_j \| \leqq K \| Q_n T y_j \|. \tag{2.23}$$

Thus

$$\begin{aligned} \| R_1 T y \| &= \left\| \sum R_1 T y_j \right\| \\ &\leqq \left(\sum \| R_1 T y_j \|^2 \right)^{1/2} \\ &\leqq K \left(\sum \| Q_n T y_j \|^2 \right)^{1/2} + \left(\sum \delta_j^2 \right)^{1/2} \\ &\leqq K c(n) \| Q_n T y \| + 2\delta_0 \\ &\leqq K c(n) \alpha + \alpha < \varepsilon / 2. \end{aligned}$$

The second line of the inequality follows from (2.1), the third line from (2.22) and (2.23), the fourth line from (2.13) and the definition of $c(n)$ and the last line from (2.12) the choice of δ_0.

The general case where we have (2.19) instead of (2.20) may be handled similarly and we leave the reader to check the details.

The construction of R_2 is identical (except for notational changes) to that of R_1. **Q.E.D.**

Remarks and Problems.

1. By an argument similar to the one above it can be shown that if X is a

complemented subspace of $(\Sigma l_q)_{l_p}$ $(1 < q < p < \infty)$ then X is isomorphic to either $(\Sigma l_q)_{l_p}$ or to a complemented subspace of $(\Sigma_{n=1}^{\infty} l_q^n)_{l_p \oplus l_q}$. This raises the following question:

PROBLEM A. *Describe the isomorphic types of complemented subspaces of* $(\Sigma l_q^n)_{l_p}$.

2. Our result lends further credence to the following conjecture:

PROBLEM B. *If X is a complemented subspace of L_p $(1 < p < \infty)$ does X have an unconditional basis?*

It was proved in [6] that such an X has a basis.

3. It is possible that Theorem 2 generalizes to subspaces of L_p. More precisely we have

PROBLEM C. *Let X be a complemented subspace of L_p $(2 < p < \infty)$ which does not contain an isomorph of Z_p. Is X isomorphic to a subspace of $l_p \oplus l_2$?*

3. An application to C_p

In this section we give an application of the above techniques to the space C_p $(1 < p < \infty)$ defined in the introduction. The isomorphic properties of C_p were discussed at some length by Arazy and Lindenstrauss in [2] and we shall not attempt to repeat all that was said there. We shall recall however certain properties of C_p which will be used in the sequel and refer the reader to [2] and the references listed there for the proofs of these properties. In what follows we assume $1 < p < \infty$.

First if we let $(e_i)_{i=1}^{\infty}$ be the unit vector basis of l_2 then if $x \in C_p$, x has a matrix representation $(x(i,j))$ where

$$x(i,j) = (xe_i, e_j) \qquad (1 \leqq i, j < \infty).$$

The operators $(u_{ij})_{i,j=1}^{\infty}$ given by

$$u_{ij}(k,l) = \delta_i^k \delta_j^l \qquad (1 \leqq i,j,k,l < \infty)$$

form a basis for C_p when suitably ordered.

Thus C_p is a matrix space with a natural basis and we shall use our above notation (e.g. O_l, $Q_{n,m}$ etc.) freely in C_p. The operators $I - O_l$, Q_n, Q^n and $Q_{n,m}$ are all norm 1 projections in C_p. Any projection $O_{n,m}$ or O_l has norm $\leqq 2$. Another natural bounded projection on C_p is the triangle projection P_T given by

$$P_T x(i,j) = \begin{cases} x(i,j) & i \geqq j \\ 0 & j > i \end{cases}$$

If S is a staircase mapping on C_p then there is a staircase mapping R so that $R(C_p) \subseteq S(C_p)$ (thus R is a "smaller" staircase than S) and such that $\|R\| \leqq M$ where M is a constant depending only on $\|P_T\|$.

T_p denotes the range of P_T and T_p is known to be isomorphic to C_p. We shall often find it convenient below for notational purposes to work with T_p rather than C_p.

$S_p = (\Sigma C_p^n)_{l_p}$ where C_p^n is the range of $(I - O_n)$. An argument similar to those of propositions 1 and 3 and lemma 2 of [2] shows that if S is a staircase mapping on T_p then there is a staircase mapping R on T_p with $RT_p \subseteq ST_p$ and such that $\operatorname{Ker} R$ is isomorphic to $l_2 \oplus S_p$.

If $E_n = O_{n,n+1}Z_p$ for $n = 0, 1, 2, \cdots$, then (E_n) is an unconditional finite dimensional decomposition of C_p. Also C_p is uniformly convex and hence super-reflexive.

Finally there is a constant $K_p < \infty$ so that if (y_i) is a norm 1 sequence in C_p with $(y_i) \in O_{n_i, n_{i+1}}$ for some increasing sequence of integers (n_i) then

$$\left\| \sum a_i y_i \right\| \leqq K_p \left(\sum |a_i|^2 \right)^{1/2} \qquad (p > 2). \tag{3.1}$$

In [2] it was shown that there are at least 9 mutually non-isomorphic complemented subspaces of C_p and the question was raised as to whether or not these are the only possible isomorphic types of complemented subspaces of C_p. The list as given there is: l_2, l_p, $l_2 \oplus l_p$, Z_p, S_p, $S_p \oplus l_2$, $S_p \oplus Z_p$, $(\Sigma Q_n C_p)_{l_p}$, C_p.

Our next theorem gives a partial answer to the above question for those complemented subspaces which do not contain an isomorph of Z_p.

THEOREM 3. *Let X be a complemented subpace of C_p $(1 < p < \infty)$ such that X contains no isomorph of Z_p. Then X is isomorphic to a complemented subspace of $S_p \oplus l_2$.*

Our first lemma is similar in statement to lemma 2.5 of [1]. The proof uses an averaging argument. For convenience we work with T_p rather than C_p.

LEMMA 5. *Let Y be a subspace of T_p $(p > 2)$ such that Y is isomorphic to l_2. Then for all $\delta > 0$ there is an integer n and an infinite dimensional subspace Z of Y so that if $z \in Z$ then*

$$\|Q^n z\| \leqq \delta \|z\|. \tag{3.2}$$

PROOF. Let $\delta > 0$ and let (y_i) be a normalized basis for Y which is C-equivalent to the unit vector basis of l_2. By passing to a subsequence we may assume that there are integers $m_1 < m_2 < \cdots$ so that for all i,

$$y_i \in O_{m_i, m_{i+1}} T_p .$$

We may also assume that

$$\lim_{i \to \infty} \| Q_{k,l} y_i \| \quad \text{exists for all} \quad k < l. \tag{3.3}$$

Let $\varepsilon > 0$. We claim there is an integer $k = k(\varepsilon)$ so that if $l > k$, then $\| Q_{k,l} y_i \| < \varepsilon$ for all but a finite number of i. Indeed if this is false then for all k there is an $l > k$ so that $\| Q_{k,l} y_i \| \geqq \varepsilon$ for an infinite number of i and thus by (3.3) for all but a finite number of i. In particular there are integers $k_1 < l_1 < k_2 < l_2 < \cdots$ so that for each j,

$$\| Q_{k_j, l_j} y_i \| \geqq \varepsilon$$

for all but a finite number of i.

Let r be any integer and choose y_i so that

$$\| Q_{k_j, l_j} y_i \| \geqq \varepsilon \quad \text{for} \quad j \leqq r. \tag{3.4}$$

Now $(Q_{k_j, k_{j+1}} y_i)_{j=1}^r$ is a (finite) montone basic sequence in C_p and C_p is super-reflexive. Thus by a theorem of James [5] there is an $\eta > 0$ and an integer s (η and s depend only on p) such that

$$\begin{aligned} 1 = \| y_i \| &\geqq \left\| \sum_{j=1}^r Q_{k_j, k_{j+1}} y_i \right\| \\ &\geqq \eta \left(\sum_{j=1}^r \| Q_{k_j, k_{j+1}} y_i \|^s \right)^{1/s} \\ &\geqq \eta \left(\sum_{j=1}^r \| Q_{k_j, l_j} y_i \|^s \right)^{1/s} \\ &\geqq \eta \varepsilon r^{1/s} \qquad \text{(by (3.4))}. \end{aligned}$$

If r is large enough we have a contradiction and the claim is proved.

Let $\varepsilon > 0$ be such that $K_p C \varepsilon < \delta$ and let $a > 0$ be such that

$$K_p C \alpha < \varepsilon / 2 \tag{3.5}$$

and choose a sequence of positive numbers $(\alpha_n)_{n=1}^\infty$ satisfying

$$\sum \alpha_n < \alpha . \tag{3.6}$$

We shall choose a subsequence (y'_i) of (y_i) and an increasing sequence of integers (k_j) satisfying the conditions:

$$Q^{k_{i+1}} y'_i = 0 \qquad \text{for all} \quad i \tag{3.7}$$

$$\| Q_{k_j, k_{j+1}} y'_i \| < \alpha_j \quad \text{if} \quad i > j. \tag{3.8}$$

To do this let $y'_1 = y_1$ and let $k_1 = k(\alpha_1)$. Let $k_2 \geqq \max(k(\alpha_2), k_1)$ be such that $Q^{k_2} y_1 = 0$ and let

$$L_1 = \{ i : \| Q_{k_1, k_2} y_i \| < \alpha_1 \}.$$

By the above claim, $N \setminus L_1$ is a finite set. Choose $y'_2 \in (y_i)_{i \in L_1}$ and let $k_3 \geqq \max(k_2, k(\alpha_3))$ be such that $Q^{k_3} y'_2 = 0$. Again by the claim if

$$L_2 = \{ i : \| Q_{k_2, k_3} y_i \| < \alpha_2 \},$$

the set $N \setminus L_2$ is finite and we choose

$$y'_3 \in (y_i)_{i \in L_2 \cap L_1}.$$

In this manner we obtain the desired sequence (y'_i).

By taking long averages we shall produce a normalized block basis (z_i) of (y'_i) such that

$$\| Q^{k_1} z_i \| \leqq \varepsilon \quad \text{for all} \quad i. \tag{3.9}$$

This is sufficient to prove the lemma for if $z = \sum \alpha_i z_i$, then

$$\begin{aligned} \| Q^{k_1} z \| &= \left\| \sum \alpha_i Q^{k_1} z_i \right\| \\ &\leqq K_p \left(\sum | \alpha_i |^2 \| Q^{k_1} z_i \|^2 \right)^{1/2} \\ &\leqq K_p \varepsilon \left(\sum | \alpha_i |^2 \right)^{1/2} && \text{(by (3.9))} \\ &\leqq K_p \varepsilon C \| z \| \\ &\leqq \delta \| z \| && \text{(by (3.5))}. \end{aligned}$$

To do this let n be such that

$$C n^{1/p - 1/2} < \varepsilon / 2 \tag{3.10}$$

and define for $i \geqq 0$,

$$z_i = \left(\sum_{j = ni+1}^{n(i+1)-1} y'_j \right) \Big/ \left\| \sum_{j = ni+1}^{n(i+1)-1} y'_j \right\|.$$

We must show that $\| Q^{k_1} z_i \| \leqq \varepsilon$ for all i and for simplicity of notation we shall assume $i = 0$ (the general case is identical).

Now,

$$\| Q^{k_1} z_0 \| \left\| \sum_1^n y_i' \right\| = \left\| Q^{k_1} \left(\sum_1^n y_i' \right) \right\|$$

(3.11)

$$= \left\| \sum_{i=1}^n Q_{k_i, k_{i+1}} y_i' + \sum_{i=1}^{n-1} Q_{k_i, k_{i+1}} \left(\sum_{j=i+1}^n y_j' \right) \right\|.$$

But

$$\left\| \sum_{i=1}^n Q_{k_i, k_{i+1}} y_i' \right\| = \left(\sum_{i=1}^n \| Q_{k_i, k_{i+1}} y_i' \|^p \right)^{1/p} \leqq n^{1/p}. \tag{3.12}$$

Also

$$\left\| \sum_{i=1}^{n-1} Q_{k_i, k_{i+1}} \left(\sum_{j=i+1}^n y_j' \right) \right\| \leqq \sum_{i=1}^{n-1} \left\| Q_{k_i, k_{i+1}} \left(\sum_{j=i+1}^n y_j' \right) \right\|$$

(3.13)

$$\leqq \sum_{i=1}^{n-1} K_p \left(\sum_{j=i+1}^n \| Q_{k_i, k_{i+1}} y_j' \|^2 \right)^{1/2} \qquad \text{by (3.1)}$$

$$\leqq \sum_{i=1}^{n-1} K_p \alpha_i n^{1/2} \qquad \text{by (3.8)}.$$

If we put (3.11), (3.12) and (3.13) together and use the fact that $\| \Sigma_1^n y_i' \| \geqq C^{-1} n^{1/2}$ we get

$$\| Q^{k_1} z_0 \| \leqq C n^{-1/2} (n^{1/p} + \alpha K_p n^{1/2}) < \varepsilon.$$

Q.E.D.

Our next lemma is similar to Lemma 1.

LEMMA 6. *Let X be a subspace of T_p $(2 < p < \infty)$ and assume that for all n and K there is a subspace $Y \subset X$ so that Y is isomorphic to l_2 and $\| Q_n y \| \leqq 1/K \| y \|$ for all $y \in Y$. Then X contains an isomorph of Z_p.*

PROOF. If Y is a Hilbert subspace of T_p then there is a $Y' \subseteq Y$ so that Y' is $2K_p$-isomorphic to l_2. The proof of this statement is identical to the proof of theorem 3.1 in [9]. By lemma 2.2 of [4] there is a normalized basic sequence (y_i) in Y such that $\| \Sigma a_i y_i \| \geqq \frac{1}{2} (\Sigma | a_i |^2)^{1/2}$, and the result follows from (3.1).

Let $\varepsilon > 0$. By the hypothesis on X and Lemma 5 there are Hilbert subspaces $W_n \subseteq T_p$, a sequence of integers $k_1 < k_2 < \cdots$ so that for all $w \in W_n$

$$Q_{k_n} w + Q^{k_{n+1}} w = 0,$$

and a subspace Z of X which is $(1+\varepsilon)$-isomorphic to $W=[W_n]$. We may assume by our initial remark that each W_n has a normalized basis $(w_i^n)_{i=1}^\infty$ which is $2K_p$-equivalent to the unit vector basis of l_2 and such that

$$w_i^n \in O_{l_i^n, l_{i+1}^n} \tag{3.14}$$

for some increasing sequence of integers $(l_i^n)_{i=1}^\infty$.

By passing to subsequences using a diagonal procedure we may also assume that if $n \neq m$ and i and j are given then there is an integer l such that

$$O_l w_i^n = w_i^n \quad \text{and} \quad O_l w_j^m = 0 \tag{3.15}$$

$$(\text{or} \quad O_l w_j^m = w_j^m \quad \text{and} \quad O_l w_i^n = 0).$$

Thus by (3.14) and (3.15) (see p. 85 of [2]) if $w_n \in W_n$ for all n then

$$\left\| \sum w_n \right\| = \left(\sum \| w_n \|^p \right)^{1/p}.$$

This shows that W and hence Z is isomorphic to Z_p. **Q.E.D.**

Proof of Theorem 3. Let X be a subspace of T_p which does not contain an isomorph of Z_p and let U be a projection of T_p onto X. First we assume $p>2$ and let $\varepsilon>0$. By Lemma 6, lemma 1 of [2] and the proof of Theorem 2 above there is a staircase mapping R and an integer n so that if $x \in Q^n T_p$ then

$$\| RUx \| \leqq \varepsilon \| x \|.$$

As we mentioned above R may be taken to have norm smaller than some constant which depends only on p and such that Ker R is isomorphic to $S_p \oplus l_2$.

The same argument used in the proof of Theorem 1 from Theorem 2 shows that there is an isomorphism S of T_p onto T_p so that $S(X)$ is contained in Ker R which is isomorphic to $l_2 \oplus S_p$.

If $1<p<2$ the result follows by duality. Indeed if X contains no subspace isomorphic to Z_p then X^* contains no isomorph of Z_q $(1/p+1/q=1)$ (we leave this to the reader). Thus X^* is isomorphic to a complemented subspace of $S_q \oplus l_2$ and so X is isomorphic to a complemented subspace of $S_p \oplus l_2$. **Q.E.D.**

References

1. D. Alspach, P. Enflo and E. Odell, *On the structure of separable $\mathcal{L}_p$ spaces* $(1<p<\infty)$ (to appear).

2. J. Arazy and J. Lindenstrauss, *Some linear topological properties of the spaces C_p of operators on Hilbert space*, Compositio Math. **30** (1975), 81–111.

3. I. S. Edelstein and P. Wojtaszczyk, *On projections and unconditional bases in direct sums of Banach spaces* (to appear).

4. P. Enflo and H. P. Rosenthal, *Some results concerning $L^p(\mu)$-spaces.* J. Functional Analysis **14** (1973), 325–348.

5. R. C. James, *Super-reflexive spaces with bases*, Pacific J. Math. **41** (1972), 409–419.

6. W. B. Johnson, H. P. Rosenthal and M. Zippin, *On bases, finite dimensional decompositions and weaker structures in Banach spaces*, Israel J. Math. **9** (1971), 488–506.

7. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, Lecture Notes in Mathematics No. 338, Springer-Verlag, 1973.

8. A. Pelczynski, *Projections in certain Banach spaces*, Studia Math. **19** (1960), 209–228.

9. A Pelczynski and H. P. Rosenthal, *Localization techniques in L^p spaces*, Studia Math. **52** (1975), 263–289.

10. H. P. Rosenthal, *On the subspaces of L^p $(p > 2)$ spanned by sequences of independent random variables*, Israel J. Math. **8** (1970), 273–303.

11. G. Schechtman, *Examples of $\mathcal{L}_p$ spaces* $(1 < p \neq 2 < \infty)$, Israel J. Math. **22** (1975), 138–147.

12. G. Schechtman, *Complemented subspaces of $(l_2 \oplus l_2 \oplus \cdots)_p$ $(1 < p < \infty)$ with an unconditional basis*, Israel J. Math. **20** (1975), 351–358.

YALE UNIVERSITY
NEW HAVEN CONN., U.S.A.

AND

OHIO STATE UNIVERSITY
COLUMBUS, OHIO, U.S.A.